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THE EXPONENTIAL SUM $C_{P,S}(r,n)$

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ABSTRACT

In this paper we introduce an exponential sum $C_{P,S}(r,n)$ as a generalization of Ramanujan's sum $C(r,n)$ and study some of its properties which give the well known result on $C(r,n)$ as particular case.

KEYWORDS: Analytic Number Theory, Arithmetical Function

1. INTRODUCTION

A real or complex valued function f defined on N is called an arithmetic function. We denote the class of all arithmetic function by A.

For any integer $n > 1$, let S_n be a nonempty set of positive divisors of n. For $f, g \in A$ their *S-convolution*, $f\,\overline{S}\,g$, is defined by

$$\left(f\,\overline{S}\,g\right)(n) = \sum_{d \in S_n} f(d) g\left(\frac{n}{d}\right),$$

Where the sum is over all elements *d* of S_n.

Observe that if $S_n = D_n$ (the set of all positive divisors of *n*) then $\left(f\,\overline{D} g\right)(n) = (f * g)(n)$, where $*$ is the classical Dirichlet convolution. Also if $S_n = U_n$ {the set of all unitary divisors of *n (*recall *d* is a unitary divisor of *n* if $d|n$ and $\gcd\left(d, \frac{n}{d}\right) = 1$)} we have $\left(f\,\overline{U} g\right)(n) = (f \circ g)(n)$, where $\circ$ is the unitary convolution studied by Eckford Cohen[3].

Introducing *S*-convolutions, Narkiewicz [6] has obtained a set of necessary and sufficient conditions on the set S_n, so that the following hold:

- $\left(A, +, \overline{S}\right)$ is a commutative ring with unity (in which ε given by $\varepsilon(n) = 1\, or\, 0$ according as $n = 1$ *or* $n > 1$ is the unity);
- $f\,\overline{S}\,g$ is multiplicative whenever *f* and *g* are;
- The arithmetic function $u(n) = 1$ for all *n* has inverse $\mu_s \in A$ relative to $\overline{S}$ (that is, $u\,\overline{S}\,\mu_s = \varepsilon$) and

$\mu_s(n) = 0 \ or \ -1$ when n is a prime power. μ_s is called the S-analogue of the Mobius function μ.

Such a S-convolution is called a *regular convolution.* Note that both Dirichlet

convolution and unitary convolution are regular.

A non empty set P of positive integers is called a *direct factor set* if for $n_1, \ n_2$ with $\gcd(n_1, n_2) = 1$ we have $n_1 n_2 \in P \Leftrightarrow n_1 \in P \, and \, n_2 \in P$. A pair P and Q of direct factor sets is said to form a *conjugate pair* if every positive integer n can be written uniquely as $n = ab$, where $a \in P$ and $b \in Q$. For such a pair note that $P \cap Q = \{1\}$. As examples of conjugate pairs of direct factor sets we have (i) $P = \{1\}$, $Q = N$ (set of all natural numbers) and (ii) $P =$ the set of all k-free integers (that is, the integers not divisible by the k^{th} power of any prime) and $Q =$ the set of all k^{th} power of positive integers.

For any integer $n > 1$, S_n denotes a set of positive divisors of n. The elements of a complete residue system (mod n) such that $(a, n)_S \in P$ where $(a, n)_S$ denotes the greatest divisor of a in S_n, is called a (P, S) *-reduced residue system (mod n)* or simply a (P, S) - *system (mod n).* A (P, S) - system (mod n) from the numbers 1,2,3,…,n will be called a *minimal* (P, S) - *system (mod n).*

In case $S_n = D_n$, the set of all positive divisors of n, we note that a (P, S)-system (mod$n$)is the P-reduced residue system (mod$n$) defined by Eckford Cohen[2].

The number of elements in a (P, S)-system (mod$n$) is denoted by $\varphi_{P,S}(n)$ and it is called the (P, S) *-totient function.* Further it may be observed that in the case $P = \{1\}$ the totient $\varphi_{P,S}(n)$ reduces to $\varphi_S(n)$, the S-analogue of the Euler totient function discussed by P. J. McCarthy[5 and others.

P.J. McCarthy[5] has defined the S-analogue of the Ramanujan sum by

$$C_S(r, n) = \sum_{(a, n)_S = 1} e^{2\pi i a / n}$$

Where the sum is over the $\varphi_S(n)$ integers a with $1 \le a \le n$ and $(a, n)_S = 1$. Among the several properties of the sum $C_S(r, n)$ the following has been obtained by P. J. Mc Carthy[4]

$$C_S(r, n) = \frac{\varphi_S(n)\mu_S(m)}{\varphi_S(m)}$$

Where $m = \dfrac{n}{(r, n)_S}$ and S is a regular arithmetic convolution.

Definition: The exponential sum $C_{P,S}(r, n)$ is defined by

- $C_{P,S}(r,n)=\sum_{(x,n)_S\in P} e(x^r,n)$

Where $e(a,n)=e^{2\pi ia/n}$ and the summation is over the integers x in a (P,S)-system $(\bmod n)$.

Note that

- $C_{P,D}(r,n)=C_P(r,n)$

Where $C_P(r,n)$ is the exponential sum introduced by Eckford Cohen[2].

Also $C_{P,S}(r,n)$ reduces to the S-analogue $C_S(r,n)$ if $P=\{1\}$

It has been noted that by Eckford Cohen [2] that in the case $P=\{1\}$ the sum $C_P(r,n)$ reduces to the well known Ramanujan trigonometric sum $C(r,n)$.

2. PROPERTIES OF $C_{P,S}(r,n)$

In this section using the methods of Eckford Cohen [2] we obtain several properties of $C_{P,S}(r,n)$ To do this we need the following result proved by V. Siva Rama Prasad and M. Ganeshwar Rao ([7] Equation 3.2).

2.1 Lemma

If P and Q form a conjugate pair of direct factor sets and S_n is a set of regular divisors of n then

$$\mu_{P,S}(n)=\sum_{d\in S_n\cap P}\mu_S\left(\frac{n}{d}\right) \tag{2.2}$$

Where μ_S is the S-analogue of μ

Also they have established a generalized inversion formula given below ([7], Theorem 4.1)

Let P, Q *be a conjugate pair of direct factor sets and* S_n *be a set of regular divisors of* n *Then for* $f,g\in A$

$$g(n)=\sum_{d\in S_n\cap Q} f\left(\frac{n}{d}\right)\Leftrightarrow f(n)=\sum_{d\in S_n} g(d)\mu_{P,S}\left(\frac{n}{d}\right). \tag{2.3}$$

Now we prove

2.4 Theorem

$$C_{P,S}(r,n)=\sum_{\substack{d\in S_n\\ d\mid r}} d\mu_{P,S}\left(\frac{n}{d}\right)$$

Proof: Let $\eta_S(r,n)=C_{N,S}(r,n)$,

Where N is the set of all natural numbers. Then by definition

2.5 $\eta_S(r,n)=\sum_{(x,n)_S\in N} e(xr,n)$

Where the sum ranges over all integers x in a (N,S) -system $(\mathrm{mod}\, n)$.

But a (N,S) -system $(\mathrm{mod}\, n)$ is a complete residue system $(\mathrm{mod}\, n)$ and therefore the sum on the right of the statement of the theorem is over a complete residue system $(\mathrm{mod}\, n)$ and therefore we have

2.6 $\eta_S(r,n)=\begin{cases} n \text{ if } n \mid r \\ 0 \text{ if } n \nmid r \end{cases}$

In view of Lemma 2.1 and (2.5)

$$\eta_S(r,n)=\sum_{d\in S_n\cap Q}\ \sum_{\left(x,\frac{n}{d}\right)_S\in P} e(dxr,n)$$

$$=\sum_{d\in S_n\cap Q} C_{P,S}\left(r,\frac{n}{d}\right)$$

Now

$$C_{P,S}(r,n)=\sum_{d\in S_n}\eta_S(r,d)\mu_{P,S}\left(\frac{n}{d}\right)$$

$$=\sum_{\substack{d\in S_n\\ d\mid r}} d\mu_{P,S}\left(\frac{n}{d}\right),$$ proving the theorem.

2.7. Corollary

- If $r\equiv 0(\mathrm{mod}\, n)$ then $C_{P,S}(r,n)=\varphi_{P,S}(n)$
- If $(r,n)=1$ then $C_{P,S}(r,n)=\mu_{P,S}(n)$

Proof: (i) If $r\equiv 0(\mathrm{mod}\, n)$ then every $d\in S_n$ is such that $d\mid r$ so that in this case Theorem 2.4 gives

$$C_{P,S}(r,n)=\sum_{\substack{d\in S_n\\ d\mid r}} d\mu_{P,S}\left(\frac{n}{d}\right)$$

$$=\sum_{d\in S_n} d\mu_{P,S}\left(\frac{n}{d}\right)$$

$$=\varphi_{P,S}(n)$$

- If $(r,n)=1$ then only $d \in S_n$ with $d \mid r$ is given by $d=1$. So that the only term on the right of the identity in Theorem 2.4 is $1 \cdot \mu_{P,S}\left(\frac{n}{1}\right) = \mu_{P,S}(n)$.

  Hence $C_{P,S}(r,n) = \mu_{P,S}(n)$ in the case $(r,n)=1$.

2.8 Remark

Corollary 2.7 shows that both $\varphi_{P,S}(n)$ and $\mu_{P,S}(n)$ are particular cases of the exponential sum $C_{P,S}(r,n)$,

It has been proved by Narkiewcz [6] that

2.9 A S-convolution is regular if and only if the sets S_n have the following property. $S_{mn} = S_m S_n = \{ab : a \in S_m, b \in S_n\}$ whenever $\gcd(r,n)=1$.

Also to prove the next theorem we need the following proved in [7]

2.10 P is a direct factor set and S_n is a set of regular divisor of n then $\mu_{P,S}$ is multiplicative.

2.11 Theorem

The function $C_{P,S}(r,n)$ is multiplicative in n.

Proof: Suppose $\gcd(m,n)=1$ then by Theorem 2.4, (2.9) and (2.10)

$$C_{P,S}(r,mn) = \sum_{d \in S_{mn}} d\mu_{P,S}\left(\frac{mn}{d}\right)$$

$$= \sum_{\substack{d_1 \in S_m \\ d_2 \in S_n \\ d_1 \mid r \\ d_2 \mid r \\ (d_1,d_2)=1}} d_1 d_2 \mu_{P,S}\left(\frac{mn}{d_1 d_2}\right)$$

$$= \sum_{\substack{d_1 \in S_m \\ d_2 \in S_n \\ d_1 \mid r \\ d_2 \mid r \\ (d_1,d_2)=1}} d_1 d_2 \mu_{P,S}\left(\frac{m}{d_1}\right)\mu_{P,S}\left(\frac{n}{d_2}\right)$$

$$= \left(\sum_{\substack{d_1 \in S_m \\ d_1 \mid r}} d_1 \mu_{P,S}\left(\frac{m}{d_1}\right)\right)\left(\sum_{\substack{d_2 \in S_n \\ d_2 \mid r}} d_2 \mu_{P,S}\left(\frac{n}{d_2}\right)\right)$$

$= C_{P,S}(r,m) \cdot C_{P,S}(r,n)$, proving the Theorem 2.11

Now Theorem 2.4, corollary 2.7 (i) and Theorem 2.4 give

2.12 $\varphi_{P,S}(n) = \sum_{d \in Sn} d\mu_{P,S}\left(\frac{n}{d}\right)$

And

2.13 $\varphi_{P,S}(n)$ **is multiplicative function.**

2.14. Theorem

$$C_{P,S}(r,n) = \frac{\varphi_{P,S}(n)\mu_{P,S}(m)}{\varphi_{P,S}(m)}$$

where $m = \frac{n}{(r,n)_S}$

Proof

Suppose $a = (r,n)_s$ then a is the greatest divisor of r which is in S_n and therefore we can write $n = a.m$ where $(a,m) = 1$. Now by Theorem 2.4 we get

$$C_{P,S}(r,n) = \sum_{d \in S_a} d\mu_{P,S}\left(\frac{n}{d}\right)$$

$$= \sum_{\substack{d\delta = a \\ d \in S_a}} d\mu_{P,S}\left(\frac{am}{d}\right)$$

$$= \sum_{\substack{d\delta = a \\ d \in S_a}} d\mu_{P,S}(\delta n)$$

Since $(a,m) = 1$ and $d\delta = a$ we get $(\delta, m) = 1$ so that by Lemma 2.1 and (2.12) we get

2.15 $C_{P,S}(r,n) = \sum_{\substack{d\delta = a \\ d \in S_a}} d\mu_{P,S}(\delta)\mu_{P,S}(m)$

$$= \mu_{P,S}(m) \sum_{d \in S_a} d\mu_{P,S}\left(\frac{a}{d}\right)$$

$$= \mu_{P,S}(m)\varphi_{P,S}(a)$$

Now $n = a \cdot m$ where $(a,m) = 1$ gives

- $\varphi_{P,S}(n) = \varphi_{P,S}(a)\varphi_{P,S}(m)$, by (2.13)

From (2.16) and (2.15) the Theorem follows.

2.16. Corollary

If $\overline{S}$ is a regular convolution then

$$C_S(r,n) = \frac{\varphi_S(n)\mu_S(m)}{\varphi_S(m)}, \text{ where } m = \frac{n}{(r,n)_S}$$

Proof: If $P = \{1\}$ in the Theorem 2.15 we get the corollary, in view of (1.3).

2.17 Remark

P.J. McCarthy [5] has proved the Corollary 2.16. It may be noted that Suryanarayana [8] has established the formula given in Corollary 2.16 in the case $\overline{S} = \overline{U}$, the unitary convolution, while the result in the case $\overline{S} = \overline{D}$ the Dirichlet convolution is well known.

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